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VOL. XIX BATON ROUGE, LA., December, 1944 No. 3

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THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer.

Papers intended for a particular Department should be sent to the Chairman of that Department.

A New Administrative Centre for The Magazine

Readers of this issue will note on the title page that our main editorial centre has been transferred from Baton Rouge, Louisiana, to Mobile, Alabama. The new adress, to which all correspondence relating to the Magazine should be sent, is

1404 E. Linwood Drive
Mobile 8, Alabama.

This address will take the place of the former one, namely, P.O. Box 1322, Baton Rouge, La., as long as it shall be impossible for us to secure a P. O. Box in the Mobile postoffice—at present an indefinite prospect.

Until further notice National Mathematics Magazine will continue to be printed by the Franklin Press, of Baton Rouge, Louisiana, and to be mailed to all subscribers from the Baton Rouge postoffice, exactly as it has been in the past.

In connection with the last statement, however, it should be **carefully noted** that our contract with Franklin Press places no further burden of distribution on them than the **one** bulk mailing which delivers to the Baton Rouge Postoffice all subscribers' copies of the Magazine, packaged by States. This means that readers who miss receiving their copies from the collective Baton Rouge mailing should address complaints or inquiries to the Editor-and-Manager at the new address in Mobile. Single or sample copies will be mailed from the Mobile office.

S. T. Sanders.

A Solid of Revolution

By James A. Ward
Delta State Teachers College

The object of this paper is to determine the equations of the median curve of the surface of revolution obtained by rotating a rectangular parallelopiped about one of its diagonals.

Let the vertex θ of the rectangular parallelopiped be at the origin of a set of euclidean space coordinates and let the adjacent vertices A, B, C have the coordinates $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ respectively. Denote the four diametrically opposite vertices by θ', A', B', C'. Set up a system of axes θU oriented from θ towards θ' and let θV be at right angles to θU in any meridian plane. Let $\Delta=\overline{oo}'=\sqrt{a^2+b^2+c^2}$. Then the values of u for the projections of θ, A, B, C, θ', A', B', C', respectively, on $\theta\theta'$ are:

$$0,\ a^2/\Delta,\ b^2/\Delta,\ c^2/\Delta,\ \Delta,\ (b^2+c^2)/\Delta,\ (a^2+c^2)/\Delta,\ (a^2+b^2)/\Delta. \tag{1}$$

Since $a/\Delta, b/\Delta, c/\Delta$ are the direction cosines of $\theta\theta'$, $(au/\Delta, bu/\Delta, cu/\Delta)$ is any point on $\theta\theta'$. The equation of the plane through this point perpendicular to $\theta\theta'$ is

$$ax+by+cz=\Delta u. \tag{2}$$

On AC', $x=a$, $y=y$, $z=0$. Hence $a^2+by=\Delta u$. Therefore the plane of equation (2) cuts AC' at

$$\left(a,\ \frac{\Delta u-a^2}{b},\ 0\right).$$

The distance from this point to $\theta\theta'$ is given by

$$V=\sqrt{\left(\frac{au}{\Delta}-a\right)^2+\left(\frac{bu}{\Delta}-\frac{\Delta u-a^2}{b}\right)^2+\left(\frac{cu}{\Delta}\right)^2}.$$

Hence

$$V^2=\frac{(a^2+c^2)u^2-2a^2\Delta u+a^2(a^2+b^2)}{b^2} \tag{3}$$

which is the median equation of the surface generated by AC' as the parallelopiped is rotated about $\theta\theta'$. It is seen that (3) is an hyperbola whose vertices are

$$\left(\frac{a^2\Delta}{a^2+c^2}, \pm\frac{ac}{\sqrt{a^2+c^2}}\right).$$

The median equations of the other edges may be found in the same manner. The six edges which do not intersect $\theta\theta'$ have hyperbolic median equations, the other six have linear median equations. All the median equations with their ranges for u are given in the table below:

ROTATED LINE	EQUATION OF MEDIAN CURVE	RANGE OF u
OA	$V=\pm\frac{\sqrt{b^2+c^2}}{a}u$	$(0, a^2/\Delta)$
OB	$V=\pm\frac{\sqrt{a^2+c^2}}{b}u$	$(0, b^2/\Delta)$
OC	$V=\pm\frac{\sqrt{a^2+b^2}}{c}u$	$(0, c^2/\Delta)$
AC'	$V^2=\frac{(a^2+c^2)u^2-2a^2\Delta u+a^2(a^2+b^2)}{b^2}$	$\left(\frac{a^2}{\Delta}, \frac{a^2+b^2}{\Delta}\right)$
AB'	$V^2=\frac{(a^2+b^2)u^2-2a^2\Delta u+a^2(a^2+c^2)}{c^2}$	$\left(\frac{a^2}{\Delta}, \frac{a^2+c^2}{\Delta}\right)$
BA'	$V^2=\frac{(a^2+b^2)u^2-2b^2\Delta u+b^2(b^2+c^2)}{c^2}$	$\left(\frac{b^2}{\Delta}, \frac{b^2+c^2}{\Delta}\right)$
BC'	$V^2=\frac{(b^2+c^2)u^2-2b^2\Delta u+b^2(a^2+b^2)}{a^2}$	$\left(\frac{b^2}{\Delta}, \frac{a^2+b^2}{\Delta}\right)$
CA'	$V^2=\frac{(a^2+c^2)u^2-2c^2\Delta u+c^2(b^2+c^2)}{b^2}$	$\left(\frac{c^2}{\Delta}, \frac{b^2+c^2}{\Delta}\right)$
CB'	$V^2=\frac{(b^2+c^2)u^2-2c^2\Delta u+c^2(a^2+c^2)}{a^2}$	$\left(\frac{c^2}{\Delta}, \frac{a^2+c^2}{\Delta}\right)$
$A'\theta'$	$V=\pm\frac{\sqrt{b^2+c^2}}{a}(\Delta-u)$	$\left(\frac{b^2+c^2}{\Delta}, \Delta\right)$

$$B'\theta' \qquad V=\pm\frac{\sqrt{a^2+c^2}}{b}(\Delta-u) \qquad \left(\frac{a^2+c^2}{\Delta},\ \Delta\right)$$

$$C'\theta' \qquad V=\pm\frac{\sqrt{a^2+b^2}}{c}(\Delta-u) \qquad \left(\frac{a^2+b^2}{\Delta},\ \Delta\right)$$

Note that the hyperbolic median curves are tangent in pairs at their common limits of range.

Assume that the notation is so chosen that $a \leqq b \leqq c$. For the present we will assume that only the inequalities hold. Then the projections (1) of the vertices of the parallelopiped and the midpoint $\Delta/2$ of $\theta\theta'$ may be arranged in increasing order:

Case 1, $c^2>a^2+b^2$.

$$O<\frac{a^2}{\Delta}<\frac{b^2}{\Delta}<\frac{a^2+b^2}{\Delta}<\frac{\Delta}{2}<\frac{c^2}{\Delta}<\frac{a^2+c^2}{\Delta}<\frac{b^2+c^2}{\Delta}<\Delta$$

Case 2, $c^2=a^2+b^2$.

$$O<\frac{a^2}{\Delta}<\frac{b^2}{\Delta}<\frac{c^2}{\Delta}=\frac{\Delta}{2}=\frac{a^2+b^2}{\Delta}<\frac{a^2+c^2}{\Delta}<\frac{b^2+c^2}{\Delta}<\Delta$$

Case 3, $c^2<a^2+b^2$

$$O<\frac{a^2}{\Delta}<\frac{b^2}{\Delta}<\frac{c^2}{\Delta}<\frac{\Delta}{2}<\frac{a^2+b^2}{\Delta}<\frac{a^2+c^2}{\Delta}<\frac{b^2+c^2}{\Delta}<\Delta$$

Let $V(\theta A)$ denote the median curve of θA, etc. By $V(\theta A)>V(\theta B)$ on (p,q) we will mean $|V(OA)| \geqq |V(OB)|$ for every u on the interval. By $V(OA)$ maximum on (p,q) we will mean $V(OA)>$ each curve on (p,q).

We are now ready to determine the median curve for the surface of revolution of the parallelopiped for Case 1. From the table it is seen that $V(OA)$, $V(OB)$, $V(OC)$ all exist on

$$\left(O,\ \frac{a^2}{\Delta}\right)$$

and that $V(OA)$ is maximum on the inteval. Over

$$\left(\frac{a^2}{\Delta},\ \frac{b^2}{\Delta}\right)$$

exist the curves $V(OB)$, $V(OC)$, $V(AB')$, $V(AC')$. Obviously $V(OB)>V(OC)$. The other two curves are tangent at a^2/Δ. At $u=0$,

$$V^2(AB') = a^4/c^2 + a^2, \quad V^2(AC') = a^4/b^2 + a^2.$$

The latter is greater, hence $V(AC') > V(AB')$ for all u. It remains to compare $V(OB)$ and $V(AC')$.

$$V^2(AC') - V^2(OB) = \frac{-2a^2\Delta u + a^2(a^2+b^2)}{b^2}$$

which vanishes for

$$u = \frac{a^2+b^2}{2\Delta}.$$

Hence $V(AC')$ is maximum over

$$\left(\frac{a^2}{\Delta}, \frac{a^2+b^2}{2\Delta}\right)$$

and $V(OB)$ is maximum over

$$\left(\frac{a^2+b^2}{2\Delta}, \frac{b^2}{\Delta}\right),$$

Over the inteval

$$\left(\frac{b^2}{\Delta}, \frac{a^2+b^2}{\Delta}\right)$$

exist $V(OC)$, $V(AB')$, $V(AC')$, $V(BA')$, $V(BC')$. The last two are tangent at b^2/Δ and $V(BC')$ is greater. $V(BC')$ and $V(AC')$ are tangent at C' and $V(BC')$ is greater. Consider $V(OC)$ and $V(BC')$.

$$\text{(4)} \qquad V^2(BC') - V^2(OC) = \frac{(c^2-a^2)\Delta^2u^2 - 2b^2c^2\Delta u + b^2c^2(a^2+b^2)}{a^2c^2}.$$

This has a negative discriminant, so cannot vanish. Therefore $V(BC')$ is maximum throughout the inteval. The curves $V(OC)$, $V(AB')$, $V(BA')$, $V(C'O')$ exist on the inteval

$$\left(\frac{a^2+b^2}{\Delta}, \frac{\Delta}{2}\right)$$

$V(C'O') > V(OC)$ on the inteval. $V(C',O')$ intersects each of the other curves beyond $\Delta/2$, hence $V(C',O')$ is maximum. Since the solid of revolution is symmetrical, it is necessary to investigate only the first half of the inteval $(0,\Delta)$.

Summarizing, we see that the median curve for Case 1 is:

ROTATED LINE	EQUATION OF MEDIAN CURVE	RANGE OF u
OA	$V=\pm\dfrac{\sqrt{b^2+c^2}}{a}\,u$	$\left(0,\ \dfrac{a^2}{\Delta}\right)$
AC'	$V^2=\dfrac{(a^2+c^2)u^2-2a^2\Delta u+a^2(a^2+b^2)}{b^2}$	$\left(\dfrac{a^2}{\Delta},\ \dfrac{a^2+b^2}{2\Delta}\right)$
OB	$V=\pm\dfrac{\sqrt{a^2+c^2}}{b}\,u$	$\left(\dfrac{a^2+b^2}{2\Delta},\ \dfrac{b^2}{\Delta}\right)$
BC'	$V^2=\dfrac{(b^2+c^2)u^2-2b^2\Delta u+b^2(a^2+b^2)}{a^2}$	$\left(\dfrac{b^2}{\Delta},\ \dfrac{a^2+b^2}{\Delta}\right)$
$C'O'$	$V=\pm\dfrac{\sqrt{a^2+b^2}}{c}\,(\Delta-u)$	$\left(\dfrac{a^2+b^2}{\Delta},\ \dfrac{\Delta}{2}\right)$

Over $\left(\dfrac{\Delta}{2},\ \Delta\right)$ the median curve is symmetrical to the above.

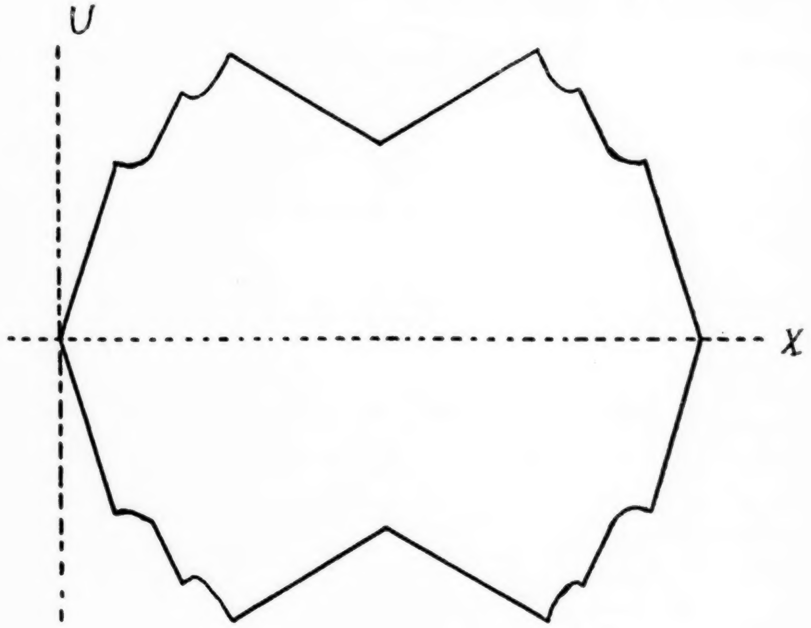


Fig 1, a=2, b=3, c=6

An illustration of Case 1 is given in Figure 1. Here is shown the median curve for the rectangular parallelopiped whose edges are in ratio 2:3:6.

In like manner it may be shown that the median curve for Case 2 is:

ROTATED LINE	EQUATION OF MEDIAN CURVE	RANGE OF u
OA	$V=\pm\dfrac{\sqrt{b^2+c^2}}{a}$	$\left(0,\ \dfrac{a^2}{\Delta}\right)$
AC'	$V^2=\dfrac{(a^2+c^2)u^2-2a^2\Delta u+a^2(a^2+b^2)}{b^2}$	$\left(\dfrac{a^2}{\Delta},\ \dfrac{a^2+b^2}{2\Delta}\right)$
OB	$V=\pm\dfrac{\sqrt{a^2+c^2}}{b}u$	$\left(\dfrac{a^2+b^2}{2\Delta},\ \dfrac{b^2}{\Delta}\right)$
BC'	$V^2=\dfrac{(b^2+c^2)u^2-2b^2\Delta u+b^2(a^2+b^2)}{a^2}$	$\left(\dfrac{b^2}{\Delta},\ \dfrac{\Delta}{2}\right)$

In Case 3 the median curve is:

ROTATED LINE	EQUATION OF MEDIAN CURVE	RANGE OF u
OA	$V=\pm\dfrac{\sqrt{b^2+c^2}}{a}u$	$\left(0,\ \dfrac{a^2}{\Delta}\right)$
AC'	$V^2=\dfrac{(a^2+c^2)u^2-2a^2\Delta u+a^2(a^2+b^2)}{b^2}$	$\left(\dfrac{a^2}{\Delta},\ \dfrac{a^2+b^2}{2\Delta}\right)$
OB	$V=\pm\dfrac{\sqrt{a^2+c^2}}{b}$	$\left(\dfrac{a^2+b^2}{2\Delta},\ \dfrac{b^2}{\Delta}\right)$
BC'	$V^2=\dfrac{(b^2+c^2)u^2-2b^2\Delta u+b^2(a^2+b^2)}{a^2}$	$\left(\dfrac{b^2}{\Delta},\ \dfrac{b^2c^2-abc\sqrt{a^2+b^2-c^2}}{(c^2-a^2)\Delta}\right)$
OC	$V=\pm\dfrac{\sqrt{a^2+b^2}}{c}u$	$\left(\dfrac{b^2c^2-abc\sqrt{a^2+b^2-c^2}}{(c^2-a^2)\Delta},\ \dfrac{c^2}{\Delta}\right)$
CB'	$V^2=\dfrac{(b^2+c^2)u^2-2c^2\Delta u+c^2(a^2+c^2)}{a^2}$	$\left(\dfrac{c^2}{\Delta},\ \dfrac{\Delta}{2}\right)$

We now consider the variations that arise by allowing two or more

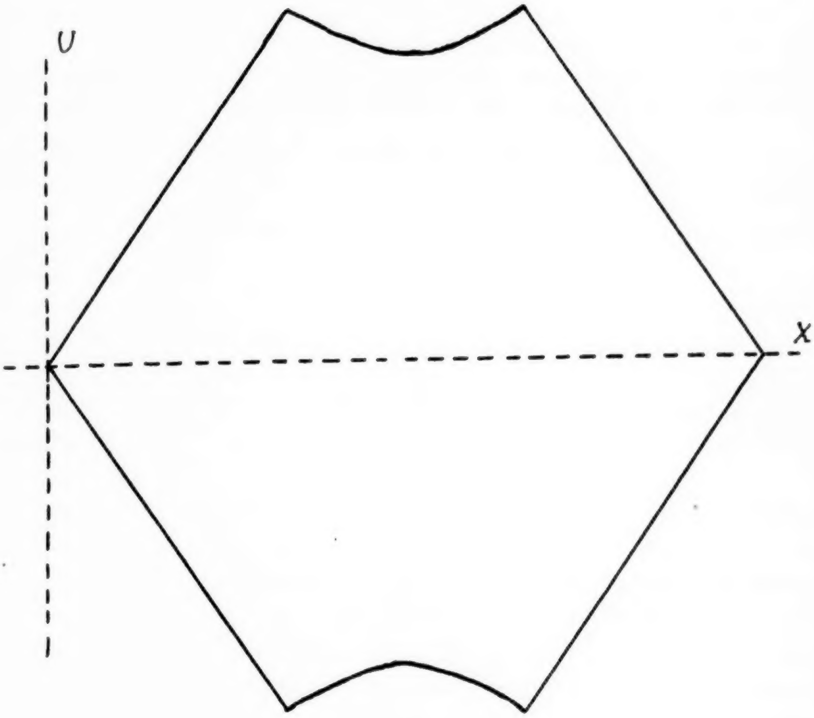


Fig 2, a=b=c

of the edges of the parallelopiped to be equal. By making $b=c$ the Case 3 becomes:

ROTATED LINE	EQUATION OF MEDIAN CURVE	RANGE OF u
OA	$V=\pm\dfrac{\sqrt{b^2+c^2}}{a}u$	$\left(0,\ \dfrac{a^2}{\Delta}\right)$
AC'	$V^2=\dfrac{(a^2+c^2)u^2-2a^2\Delta u+a^2(a^2+b^2)}{b^2}$	$\left(\dfrac{a^2}{\Delta},\ \dfrac{a^2+b^2}{2\Delta}\right)$
OB	$V=\pm\dfrac{\sqrt{a^2+c^2}}{b}u$	$\left(\dfrac{a^2+b^2}{2\Delta},\ \dfrac{b^2}{\Delta}\right)$
BC'	$V^2=\dfrac{(b^2+c^2)u^2-2b^2\Delta u+b^2(a^2+b^2)}{a^2}$	$\left(\dfrac{b^2}{\Delta},\ \dfrac{\Delta}{2}\right)$

In this case the tangent to the median curve is continuous at $\Delta/2$, which was not true of the previous cases.

Since the special case $a=b$ can occur in each of the four cases above, there are eight different types of median curves for the solid.

Figure 2 shows the median curve for a cube. The entire median curve is:

ROTATED LINE	EQUATION OF MEDIAN CURVE	RANGE OF u
OA	$V=\pm\sqrt{2}u$	$\left[0, \dfrac{a}{\sqrt{3}}\right]$
AC'	$V^2=2(u^2-a\sqrt{3}u+a^2)$	$\left[\dfrac{a}{\sqrt{3}}, \dfrac{2a}{\sqrt{3}}\right]$
$C'O'$	$V=\pm\sqrt{2}(a\sqrt{3}-u)$	$\left[\dfrac{2a}{\sqrt{3}}, a\sqrt{3}\right]$

Therefore the solid of revolution for the cube consists of three sections. The middle section is an hyperboloid of revolution of one sheet, and the end sections are conical. The volume of the middle section turns out to be $2\frac{1}{2}$ times the volume of an end section.

CORRIGENDA

Exercise No. 3, of page 36, October issue, should have read:

$$3.\ 1+3/5+4/10+5/17+\ldots\ldots 1+n/1+n^2=\frac{4n^{2(2n+109)}}{127n^2+305n+12}$$

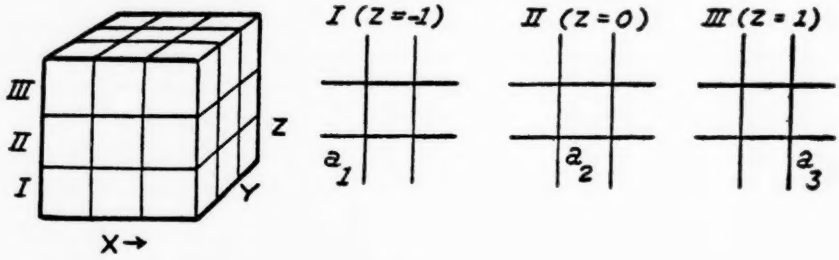
Hyper-Spacial Tit-Tat-Toe or Tit-Tat-Toe In Four Dimensions

By William Funkenbusch and Edwin Eagle
Oregon State College

Perhaps you are one of those people who found college uninteresting because the game of tit-tat-toe was too simple to while away the time in afternoon lecture courses. If so, by extending the game to three, to four, and to n dimensions, you will find sufficient mental stimulation to revive your interests.

Probably all of you have at some time played ordinary two dimensional tit-tat-toe. Some of you probably have seen the three dimensional game, which is played in a cube. For those who have not, we present the three dimensional game before going into the hyper-spacial game.

Consider a cube $3'' \times 3'' \times 3''$ cut up into 27 inch cubes. If we take the centroid of the center cube as the origin we have a range from minus one to plus one in all three directions. By imagining the three horizontal layers sliced apart and placed side by side we can represent the entire cube by three separate two dimensional drawings.



The game can be played by symmetry from the drawings, or the three points of a "win" may be proved to be collinear by showing that the direction numbers of the lines joining the points in pairs are proportional. For example, consider the win "a" shown above.

a_1 $(-1, 1, -1)$ $\qquad$ a_2 $(0, 1, 0)$ $\qquad$ a_3 $(1, 1, 1)$

We may calculate the direction numbers of

$$\text{line } a_1a_2\text{: } (-1\ -0),\ (1\ -1),\ (-1\ -0) = -1,\ 0,\ -1$$

$$\text{line } a_2a_3\text{: } (\ 0\ -1),\ (1\ -1),\ (\ 0\ -1) = -1,\ 0,\ -1$$

Therefore line a_1a_2 is parallel to a_2a_3, and therefore a_1, a_2 and a_3 are collinear points.

The two dimensional game is logically for two players; a player can sometimes set up two possible wins for his next play, thus cinching a win. Similarly, the three dimensional game is logically for three players; by skillful play it is sometimes possible to set up three possible wins, so even with two players following the win cannot be blocked. In the same way, the four dimensional game to be described is best suited for four players.

Now let us consider the hyper-spacial game. Let us develop it for a "four space," from which the extension to an "n space" is obvious, though the geometric representation becomes increasingly difficult as n increases. We can give the four dimensional game physical significance by using a physical property such as temperature, potential, or intensity of color as the fourth dimension. If we use color, blue might represent plus one; green, zero; and red, minus one. To secure a significant spacial relationship we can imagine the 27 cubes in the $3\times3\times3$ arrangement each sliced into three horizontal sections, the top section being blue, the middle one green, and the bottom one red. As before we can imagine the layers I, II, and III layed out to give us three separate drawings, each of which would represent a $3\times3\times1$ prism with the three sections of different colors. Now if we think of the blue section of each sliced off and placed above, and the red section of each sliced off and placed below, we have our $3\times3\times3\times3$ hyper-space represented as nine separate two dimensional drawings.

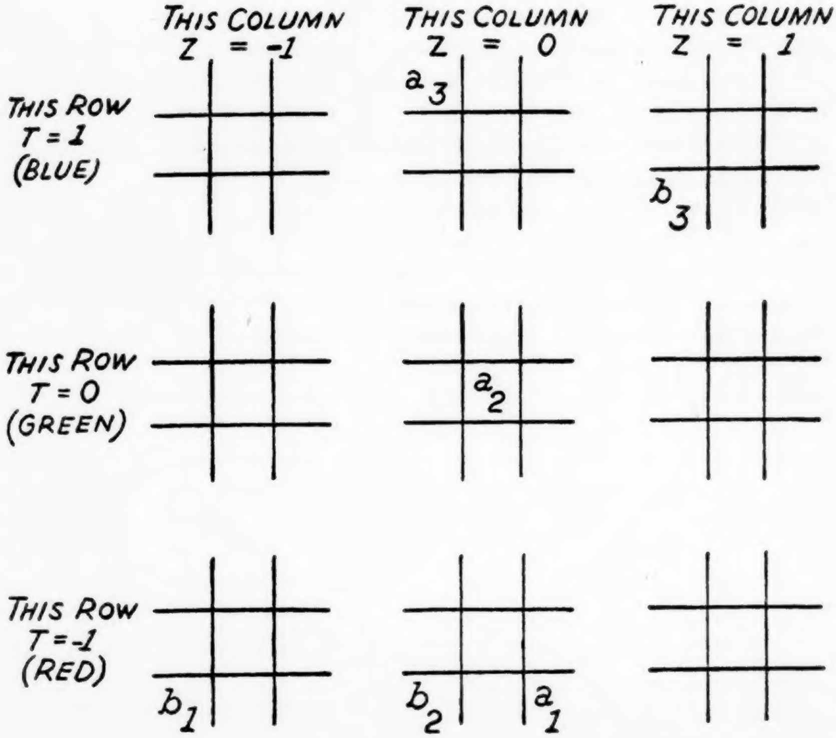
A line in four space will have four direction numbers, and we shall say that the points are collinear and represent a win if the direction numbers of the lines joining the points in pairs are proportional. Again, the game may be played by symmetry or by the direction numbers of the lines. The win "a" above is:

$$a_1\ (1, 1, 0, -1) \qquad a_2\ (0, 0, 0, 0) \qquad a_3\ (-1, -1, 0, 1)$$

Direction numbers of line a_1a_2 are $1, 1, 0, -1$

Direction numbers of line a_2a_3 are $1, 1, 0, -1$

Since their directions are the same and since a_2 is common, the line segments are parts of the same line and the three points are collinear.



With this scheme for representing the four dimensional game it will be noticed that every win gives three points which are collinear in the three dimensional sense. However, there are some combinations collinear in the three dimensional sense which are not wins in the four dimensional game. For instance, imagine in the solid cube with the three color scheme, a vertical line in the front left hand corner connecting three points, b_1 in the red section of the bottom cube, b_2 in the red section of the middle cube, and b_3 in the blue section of the top cube.

$b_1\ (-1, 1, -1, -1) \qquad b_2\ (-1, 1, 0, -1) \qquad b_3\ (-1, 1, 1, 1)$

Direction numbers of b_1b_2 are 0, 0, 1, 0.

Direction numbers of b_2b_3 are 0, 0, 1, 2.

Hence the points are not collinear in the four dimensional sense. Evidently in a four dimensional win, not only are the points collinear in the three dimensional sense, but also the three dimensional representations of b_1b_2 and b_2b_3 are equal in length.

It is apparent that the game may be complicated in two differ-

ent ways: (1) by increasing the extent beyond plus one and minus one; (2) by increasing the number of dimensions.

The authors do not anticipate that four dimensional tit-tat-toe will become a popular pass-time. We do hope however that this scheme for representing a fourth dimension may prove interesting and stimulating. We imagine that similar representations may be helpful in problems of a more practical nature. By using a certain portion of the spectrum to represent a desired number range, or by using the entire range to represent from minus infinity to plus infinity the scheme might conceivably be adapted to a great variety of problems.

Covering Problems

BY L. M. KELLY
U. S. Coast Guard Academy, New London, Conn.

1) Covering problems seem to have received comparatively little attention in elementary mathematics. In the more sophisticated branches (e.g. theory of functions) numerous covering problems have been considered and solved. The Heine-Borel, the Lebesgue, and the Vitalli theorems are but a few of the better known results. Many of the highpowered imbedding theorems might take on added interest to the average mathematician if couched in the "covering" phraseology. That "covering" and "imbedding" are equivalent notions should be quite clear. For example if a given circular disc can "cover" a set of points, one could equally well say that the points could be "congruently imbedded" in the disc. In the first case we consider moving the disc to cover the points and in the second moving the point set to "fit in" the disc. To make clear the type of problem we propose to consider, let us pose the following questions. Suppose given a circular disc (a pie plate if you will) and a plane set of points of such a nature that each five of these points can be covered by the disc. Does it then follow that the pie plate can cover the whole set at once? If the answer is yes, then can the number be reduced from 5 to 4 or 3? If the answer is no, how about raising the 5 to 6 etc.? As a matter of fact the answer to the first question is yes, and the number can be reduced from 5 to 3. The proof of this fact is not too easy without the aid of a certain powerful theorem. Before proceeding to its consideration, we will introduce some useful terminology and definitions.

2) The fact that a circular disc may be made to cover the whole of a plane set of points if it can cover each subset of three, is described by Professors Menger and Blumenthal by saying that the disc has "congruence order three with respect to plane sets." That the disc does not have congruence order three with respect to three dimensional point sets, may be simply shown by counter examples. Assuming congruence order three with respect to plane sets, it immediately follows that the disc has congruence order (hereafter c.o.) 4 with respect to sets in E_3. The following are evident or easily proved.

a) The unlimited straight line has c.o. 3 with respect to Euclidean sets.
b) The line segment has c.o. 2 with respect to linear pt. sets.
c) The plane has c.o. 4 with respect to Euclidean sets.
d) If a figure has c.o. n it has c.o. $n+k$ where $k=1,2,3, \cdots$
We are usually interested in the minimum c.o.

To prove that the circular disc has c.o. 3 with respect to plane sets we appeal to the following theorem of Helly.*

* E. Helly, Uber Mengen konvexer Korper mit gemeinschaftlichen punkten. Jahres berichte der deutschen Math - Ver Bd 32 (1923).

"If in Euclidean n-space each $n+1$ members of a set of convex bodies have a point in common then they all have." In the two dimensional case this becomes "If each three of a set of convex discs in the plane have a point in common, then they all have." We are now in a position to prove.

Theorem 1: The circular disc has c.o. 3 with respect to plane sets.

Proof: Consider any three points of the set. Describe three circles of the given radius having these points as centers. These discs must have a point in common for otherwise the three points would not be coverable by the given disc. Hence we see that the set of all discs having for centers the given points and the radius of the given disc as radius must have a point in common by Helly's theorem. This implies that all the points of the set may be covered by the disc.

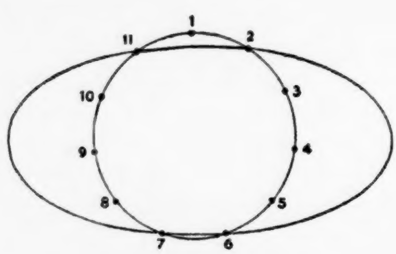


Figure 1

It would be quite natural to seek a generalization of this result in the elliptical disc. Surprisingly enough, however, there is little trouble in showing that this disc has no finite congruence order with respect to the Euclidean sets. That is to say that if we are given an elliptical disc and are guaranteed that it will cover each n points of a set (having more than n members of course) this will be no guarantee that it can cover the whole set at once. N here is any finite

number. The accompanying figure illustrates the case of an elliptical disc capable of covering each 10 points of a set but not the whole set at once. Thus the c.o. is certainly not 10. Furthermore, it is clear that this example may be extended to any finite number. One of the common methods of constructing examples of this nature is to imagine a small odd sided polygon placed inside the disc and then "blown up" till one vertex sticks out and the remainder of the figure fits "snuggly" inside. For example, suppose we have a narrow "annular ring." If we place a small regular odd sided polygon in the ring as shown (Figure 2) and then "blow it up" as previously described it becomes apparent that the ring has no finite c.o. This example is only valid for $r \geq \frac{1}{3}R$, for if r gets too small it is clear that the "blown up" polygon could be fitted around the hole of the ring. The following example (Figure 3) takes care of the case where $r < \frac{1}{3}R$. The rectangular, square and triangular "plates" can all be shown in a similar fashion to have no finite c.o. with respect to plane point sets. As a matter of fact aside from the circular case no "disc" or "plate" is known to have a finite c.o. with respect to plane sets. If this were actually so, it would provide an interesting characterization of the circular disc. Dr. Robinson has shown that it is the only disc having congruence order three.

3) We consider now certain of the more common plane curves. The circle presents a very interesting case. That the circle has c.o. 4 with respect to point sets in E_2 is at once apparent since three points determine a circle. However, this cannot be reduced to three. For consider the vertices of any triangle together with its orthocenter (an orthocentric quadruple). It is well known that the four circumcircles determined by each three of these points are equal. (N. A. Court, *College Geometry*, p. 99, art. 174.) In other words, we have a set of points each three of which are coverable by a certain circle but the whole set not. It will next be shown that if there are more than four points in the set, then the coverability of each three does imply the coverability of the whole set. Suppose (if possible) a set of 5 points, each three coverable by a given circle but the whole set not. First note that through two fixed points but two distinct circles of a given radius can be passed (in a single plane) and that these are reflections of each other in the line determined by the two points. If our five points be labeled 1, 2, 3, 4, and 5 it is seen at once that 123, 124 and 125 determine three equal circles all of which cannot be distinct. No loss of generality will result if we assume the first two to coincide, thus making 1, 2, 3, and 4 circular. Again 345, 245, and 145 likewise determine three equal circles, two of which

CASE 1

$r \geq \frac{1}{3}R$

Showing 11 points, each 10 coverable but the whole set not.

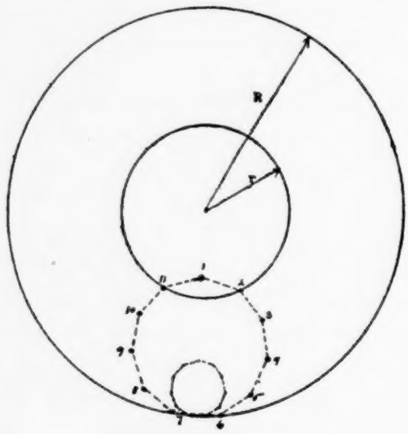


Figure 2

CASE 2

$r < \frac{1}{3}R$

Showing 11 points, each 10 coverable but the whole set not.

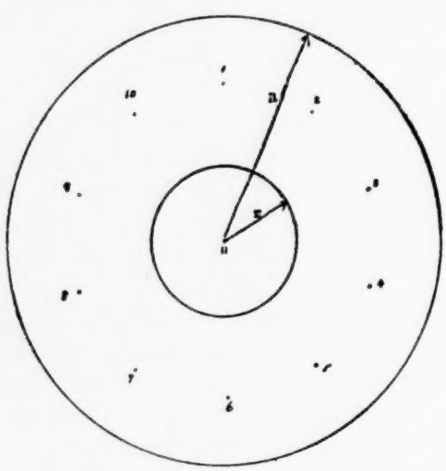


Figure 3

must coincide. Regardless of which two are selected, the resulting circle is bound to have three points in common with 1, 2, 3, 4. Thus the five points are circular contrary to the hypothesis. It can be concluded, then, that a set of points in the plane, having more than five elements, each three of which can be covered by a given circle (curve), can itself be covered by that circle. The only points, then, that stand in the way of a reduction of the c.o. of the circle from four to three are certain quadruples. This sort of thing led Blumenthal to introduce the notion of congruence indices. He would say in this instance that the circle had congruence indices (3, 1) with respect to plane sets, meaning that if a set of points in the plane having more than $3+1=4$ members, is such that each subset of three is coverable by a given circle, then the whole set is likewise. The quadruples which stand in the way of a reduction these indices are called "pseudo circular quadruples." Attention has already been called to the fact that orthocentric quadruples form such a set. The subsequent discussion is designed to show that these are the only such sets. Let A, B, C, and D form a set of this nature. The circumcircles of ABD, ACD, and BCD are all equal to the circumcircle of the triangle ABC. Their centers then, must be the three reflections of the circumcenter O of ABC in the respective sides. Moreover, D is at a distance R from these three reflections and is thus the center of the circle determined by them. But it is known that the center of such a circle is the isogonal conjugate of O, or in other words the orthocenter of ABC. We have thus completely characterized those sets of points in the plane each three of which are coverable but the whole set not.

Our attention has been confined, in this last discussion, to plane sets of points. What about three dimensional sets? Certainly the circle has c.o. 4 with respect to sets in E_3 but the four vertices of a regular tetrahedron present an obvious three dimensional pseudo circular quadruple. First we ask if this is the most general such quadruple. The answer here is in the negative since the isosceles tetrahedron has the circumcircles of all four faces equal. Furthermore, if the four circumcircles of the four faces of a tetrahedron are equal the incenter and circumcenter must coincide. This is a *n.* and *s.* condition that the tetrahedron be isosceles, (see N. A. Court, *Modern Pure Solid Geometry*, p. 97). All pseudo circular quadruples in three space (non-degenerate) are vertices of an isosceles tetrahedron. But maybe we can have pseudo quintuples. Suppose points labeled 1, 2, 3, 4, and 5 formed such a set.

1) They cannot be planar.

2) No four can be planar. For suppose 1234 were planar. The four quadruples containing 5 would each be pseudo, from which it readily follows that the quintuple would be equilateral. But there are no equilateral quintuples in E_3.
3) If no four are planar it is similarly true that the quintuple would have to be equilateral.

Our previous results can now be generalized. With respect to point sets in E_3 the circle has indices (3,1). The pseudo sets are of two types, the plane orthocentric quadruple and the vertices of isosceles tetrahedron.

This rather complete discussion of the circle gives some indication of the type of analysis desired. Other familiar curves do not yield quite so easily. The conic, for example, has not yet been completely treated. Certainly it has congruence order six, since five points determine a conic. Whether this can be lowered to 5, is still a mystery. The best that has been thus far shown is that it has indices (5,1) and that in the special case of the equilateral hyperbola the c.o. is five. We will prove these assertions. The following theorems will be used in the proofs,

1) Five points determine a conic.
2) If two equilateral hyperbolae intersect in four real points those points form an orthocentric quadruple.
3) Through four points (not an orthocentric set) it is possible to pass at most two distinct similar conics.

Let us dispose of the case of the equilateral hyperbola first.

Proof: Suppose we have six points labeled 1, 2, 3, 4, 5, and 6, each five of which are coverable by a given equilateral hyperbola but the whole set not. This would necessitate two distinct congruent equilateral hyperbolae through each four points. But by property two above each quadruple would then be orthocentric, which is clearly impossible. Hence the congruence order of the equilateral hyperbola is five.

Passing now to a consideration of the general case we fix our attention on a set of seven points labeled from 1 thru 7, each five of which are coverable but the whole set not, and proceed to show that such a situation is impossible.

Proof: Since each five are coverable, it follows that the three quintuples, 12345, 12346, and 12347 determine three congruent conics. By property three above these canot all be distinct. Suppose (with no loss of generality) that the first two coincide. Then 1, 2, 3, 4, 5, 6 are all on the same conic. Again 71234, 71235, and

71236 determine three congruent conics, two of which must coincide. No matter which two we take, the resulting conic will have five points in common with 123456 and hence all seven points will be on the same conic. Thus, if each five of a set of seven or more points can be covered by a given conic, the whole set can likewise be covered. So the only question remaining is whether there exists a set of six points each five coverable but the whole set not. Possibly some reader can supply the answer. In addition, we will propose the following as exercises.

1) Exhibit a set of five points each four of which are coverable by a conic but the whole set not.

2) Characterize such sets.

3) What is the c.o. of a circular arc with respect to point sets in E_3? There are several cases to consider.

It seems probable that some very general theorems with regard to plane curves can be developed.

4) These circular results can be extended to a certain degree to the spherical surface and, as a matter of fact, to the n- dimensional sphere. We will confine our discussion to the sphere in three-space leaving the generalization to any who might be interested. The sphere will be shown to have congruence indices, (4,1). The existence of a pseudo spherical quintuple will be shown and characterized.

Congruence order five with respect to points in E_3 is at once evident. Let us now examine a set of six points each four of which are coverable by a sphere of radius R but the whole set not. Label the points 1 thru 6. Note that thru three non collinear points there can pass but two distinct spheres of a given radius. 1234, 1235, and 1236 are all coverable and hence the three circumspheres are equal. These cannot be distinct. Suppose the first two coincide. Then 12345 are all on a sphere of radius R. Similarly 4561, 4562 and 4563 have equal circumspheres which cannot all be distinct. The coincidence of any two would imply four points in common with 12345 and thence that the six points are cospherical. The sphere, then, has indices (4,1) with respect to point sets in E_3.

The question of the existence of a pseudo quintuple arises. Howard Eves recently showed in the Am. Math. Monthly (Vol. 50, 1943, p. 389) that there exists a tetrahedron for which the radius of the sixteen point sphere is one half the circumradius. We will show that the vertices of such a tetrahedron together with the isogonal

conjugate of the circumcenter form a pseudo spherical quintuple and further that such a set is the only one having this property.

Let points *A, B, C, D,* and *E* form a pseudo spherical quintuple. The circumspheres of *ABCE, ABDE, BCDE* and *ABDE* are all reflections of the circumsphere of *ABCD* in the respective faces. Their four centers are therefore the four reflections of the circumcenter of *ABCD* in the faces. These four centers are at a distance *R* from *D* which means that *D* is the center of the circumsphere of these four reflections. It follows that *D* must be the isogonal conjugate of the circumcenter of *ABCD* (see Court, *Modern Pure Solid Geometry*, p. 244) and in addition that the radius of the sixteen point sphere is one half of *R*. EVES' proof establishes the existence of such a tetrahedron. While we have characterized pseudo-quintuples, it would be interesting to know more of the geometric properties of such a set. Apparently it has not been very extensively investigated.

This paper has been concerned with a general introduction to methods and terminology. No attempt has been made to include all known results even in Euclidean spaces. For a more detailed and advanced treatment of congruence order and related problems (1), (2), and (3) are recommended.

Dr. Robinson has several very interesting results concerning c.o. of spherical caps with respect to spherical sets.

(1) L. M. Blumenthal—Distance Geometries. University of Missouri Studies.

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Humanism and History of Mathematics

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The Influence of Mathematics on the Philosophy of Leibniz*

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1. *Introduction.* In previous papers the writer has considered the influence of mathematics on the philosophy of Descartes and of Spinoza. This paper considers the third of the great continental rationalists in philosophy, Gottfried Wilhelm Leibniz, who was also one of the greatest of modern mathematicians. After brief consideration of his life, his mathematics, and his philosophy, the writer will point out some of the ways in which mathematics influenced the philosophy of Leibniz. His philosophy will be considered from the point of view of the major problems with which he dealt.

2. *Life (1646-1716).* Gottfried Wilhelm Leibniz was born in Leipzig in 1646, the son of a professor of the University of Leipzig. He entered the university at fifteen, and at twenty had completed all requirements for a doctorate, but was refused the degree because of his youth. He received his doctorate in laws, however, from the University of Altdorf the same year, and refused to accept a professorship there.

During his life Leibniz held many public offices and received many titles, but he is best known for his work as a mathematician and a philosopher. He traveled extensively, going to Paris and London, where he knew the foremost scholars of the time. He became librarian of the House of Brunswick in 1676 and held the position during the most of the remainder of his life. George I of England came from the House of Brunswick in 1714, but he required Leibniz to remain at Hanover to keep the library. Leibniz became a member of the Royal Society of London in 1673 and the French Academy in

* **Read before the Tennessee Academy of Science, May 1, 1942.**

1700. In 1700 he was instrumental in founding the Berlin Academy of Sciences and became its first president. He died in 1716 after his last years were made unhappy by controversy with Newton's disciples.

Leibniz's most important works include: *Disputatio Metaphysica de Principio Individui* (1663), *Nouveaux Essais sur l'entendement Humain* (1703), *Essais de Théodicée* (1710), and *La Monadologie* (1714). A somewhat less well known work, the *Dissertatio de Arte Combinatoria* (1666) is important for a consideration of the influence of mathematics on his philosophy.

3. *Mathematics.* Though he is known as one of the greatest of modern mathematicians, Leibniz published no book in mathematics, only a few fragments. He learned the subject by himself, though Huyghens perhaps interested him in mathematics during one of his visits to Paris. Leibniz worked problems in mathematics throughout his life, and his knowledge of mathematics can be seen in his correspondence. His principal contribution to mathematics was the invention of the calculus. There was, however, a great controversy as to whether he got his ideas from Sir Isaac Newton. Authorities now generally agree that Leibniz invented the calculus independently of any knowledge of Newton's fluxions, though Newton had the idea of the calculus earlier than Leibniz. The notation of the calculus was undoubtedly invented by Leibniz.

Some other points which are rather minor in comparison with the calculus may be mentioned. Leibniz invented a calculating machine which would add, subtract, multiply, divide, and take roots. He invented determinants, but perhaps the Japanese mathematician Seki-Kowa had already used determinants ten years earlier. He laid the foundation for the theory of envelopes, and introduced the terms "coordinates" and "axes of coordinates." Finally, he was probably the first to use the term "function," the concept of which is considered the unifying thread of all branches of mathematics.

4. *Philosophy.* It should be remembered that in the time of Leibniz the term "philosophy" included all that there was of science. There was, however, very little of what we call *experimental* science. Leibniz was one of the greatest of modern philosophers, belonging to the school of "Continental Rationalism." Descartes had believed that his first duty was to doubt all the conclusions of his predecessors, while Leibniz was of the opinion that his first duty was to show how near all his predecessors had come to the truth. He harmonized the "mind" and "matter" of Descartes, and his philosophy may be described as a partial Idealism.

It will be sufficient here merely to list the principal tenets of the philosophy of Leibniz. The most important of these was his doctrine of *monads*. For Leibniz, substance was essentially active rather than passive, or in other words, "To be is to act." The centers of activity were the monads. The doctrine of monads bore a slight similarity to modern quantum theory. The second tenet was the *pre-established* harmony, by means of which Leibniz tried to explain the separation of mind and matter by Descartes. He declared that mind and matter were like two clocks set to run together, being synchronized by God. The third tenet was the law of *continuity*, which declared that there was neither vacuum nor break in nature, but that the different species of creatures arose by imperceptible degrees from the lowest to the most perfect form. The fourth tenet was *optimism*, the belief that the world was the best of all possible worlds.

5. *Influence of Mathematics on the Philosophy of Leibniz.* A man's philosophy may be considered in terms of his treatment of the following problems: the problem of method; epistemology: the problem of knowledge; metaphysics: the problem of ultimate reality; natural philosophy: the problem of the external world; and practical philosophy: the problems of ethics, esthetics, and politics. The following sections of this paper consider the evidence of the influence of mathematics on Leibniz's treatment of each of these problems.

6. *The Problem of Method.* The most significant effect of mathematics on Leibniz's philosophical method was his *Universal Mathematics*. He carried out this idea more fully than Descartes had and tried to interest Louis XIV of France in the use of a system of universal symbols. According to Lewis, Leibniz wrote in a letter to Galois in 1677:

> The true method should furnish us with an Ariadne's thread, that is, with a certain sensible and palpable medium, which will guide the mind as do the lines of geometry and the formulae for operations which are laid down for the learner in arithmetic.[1]

According to Child,

> . . . . the main ideas of his philosophy are to be attributed to his mathematical work, and not *vice versa*. The manuscripts of Leibnitz, which have been preserved with such great care in the Royal Library at Hanover, show perhaps more clearly than his published work, the great importance which Leibnitz attached to suitable no-

[1] C. I. Lewis and C. H. Langford, *Symbolic Logic* (New York: The Century Company, 1932), p. 6.

tation in mathematics and, it may be added, in logic generally. He was, perhaps, the earliest to realize fully and correctly the important influence of a calculus on discovery. The almost mechanical operations which we go through when we are using a calculus enable us to discover facts of mathematics or logic without any of that expenditure of the energy of thought which is so necessary when we are dealing with a department of knowledge that has not yet been reduced to the domain of operation of calculus.[2]

According to Erdmann:

Leibnitz, therefore, like Bacon, urges that facts should be collected, and thinks there can never be enough of repositories and academies, the use of which he himself compares to tables of logarithms. Once these data are procured, we must set to work with them, a process which he is very fond of calling a kind of reckoning, *calculus ratiocinator*. . . . The word, however, must be understood in such a wide sense that ordinary reckoning, as well as the ordinary syllogistic process forms only a small part of what it includes. Like all reckoning this higher calculus has two parts—association and separation, synthesis and analysis. The method of combination is an essential part of the synthetic process; by it we can, for example, calculate the possible total of all pieces of music, in fact, can find out these pieces themselves. The synthetic process tells us whether and how problems can be combined. The process of analysis, on the other hand, deals with the individual problem, breaks it up into easier ones, and if it does not solve it, at least brings us nearer a solution. . . . The explanation of Descartes must have shown Leibnitz what was in any case very obvious, that the most important point for every system of calculus is the happy choice of symbols. . . . Taking all this into consideration, we need not be surprised that, his whole life through, Leibnitz was thinking of a system of symbols, by the help of which, every primary idea could be fixed in a single formula. . . . The main point with him was that a system of symbols would be chosen, the effect of which would be that every faulty combination of thoughts would necessarily lead to an impossible or self-contradictory formula, every hiatus in reasoning necessarily show itself in a want of connection between the characters, and so on. . . . accordingly he confines himself to mathematical symbols, experimenting sometimes with lines, sometimes with figures, sometimes with letters. That he did not succeed in achieving the desired results is well known.[3]

Another minor point in regard to the influence of mathematics on the method of Leibniz may be mentioned. In his dissertation called *De Arte Combinatoria*,[4] written at an early age, Leibniz de-

[2] J. M. Child, *The Early Mathematical Manuscripts of Leibnitz* (Chicago: The Open Court Publishing Company, 1920) p. iii.

[3] J. E. Erdmann, *A History of Philosophy*, translated by W. S. Hough (New York: The Macmillan Company, 1924) II, 193.

[4] J. E. Erdmann (editor), *God. Guil. Leibnitii Opera Philosophica* (Berolini: Sumtibus G. Eichleri, 1840), pp. 6-44.

clared that simple and primitive concepts should be symbolized by prime numbers, and that the combination of two concepts should be represented by their product. Thus, if three represented "rational" and seven represented "animal," then "twenty-one" would represent "man." He used negative numbers also, but the scheme broke down and he gave up the system of representing concepts by numbers.

7. *Epistemology: the Problem of Knowledge.* The problem of epistemology is closely related to the problem of method, since a philosopher's method is for the purpose of obtaining true knowledge. The most significant point to be mentioned in connection with this problem is the fact that Leibniz was important in the historical evolution of symbolic logic, as finally worked out by Whitehead, Russell, and others. Leibniz declared that he tried to reduce logic to the same state of certainty as arithmetic. In his *New Essays on the Human Understanding,* Leibniz made Philethes say, "I begin to form for myself a wholly different idea of logic from that which I formerly had. Then I regarded it as a scholar's diversion, but now I see that, in the way you [Leibniz] understand it, it is like a universal mathematics."[5] In the same work Leibniz declared:

> There are very many examples of demonstrations outside of mathematics and one can say that Aristotle gave some of them in his first *Analytiques.* Indeed logic is just as susceptible to demonstration as geometry; and one can say that the logic of the geometers is based on the work of Euclid or a particular extension of general logic.[6]

According to Lewis and Langford:

> The program both for symbolic logic and for logistic, in anything like a clear form, was first sketched by Leibnitz, though the ideal of logistic seems to have been present as far back as Plato's *Republic.* . . . Leibniz correctly foresaw the general character which logistic was to have and the problems it would set themselves to solve. But . . . . he failed of any clear understanding of the difficulties to be met, and he contributed comparatively little to the successful working out of details. Leibnitz expected that the whole of science would shortly be reformed by the application of this method. This was a task clearly beyond the powers of any one man. . . .[7]

Bertrand Russell is one of the scholars who have brought symbolic logic to its maturity. This is what Russell had to say about Leibniz's part in the development of symbolic logic:

[5] *Ibid.*, p. 399.

[6] J. M. Bruyset (editor), *Esprit de Leibnitz, ou Recueil de Pensées Choisies* (Lyon: Jean-Marie Bruyset, 1772), II, 137.

[7] *Op. cit.*, p. 4.

He seems to have thought that the symbolic method in which formal rules obviate the necessity of thinking, could produce everywhere the same fruitful results as it has produced in the sciences of number and quantity. . . . "If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to set down to their stalls, and say to each other (with a friend as witness, if they liked): 'Let us calculate.' "[8]

Another point in Leibniz's treatment of the problem of Knowledge which may be mentioned here was his principle of *sufficient reason*. Near the end of his life Leibniz carried on a controversy with Samuel Clarke of England. In his second "Reply" to Clarke, Leibniz declared:

The great foundation of mathematics is the principle of contradiction or identity, that is, that a proposition cannot be true and false at the same time; and therefore A is A, and cannot be $-A$. This single principle is sufficient to demonstrate every part of arithmetic and geometry, that is, all mathematical principles. But in order to proceed from mathematics to physics, another principle is requisite, as I have observed in my *Théodicée*: I mean, the *principle of a sufficient reason,* viz., that nothing happens without a reason why it should be so, rather than otherwise. And therefore Archimedes, being desirous to proceed from mathematics to physics, in his book *De Aquilibrio* was obliged to make use of a particular case of the great principle of a sufficient reason.[9]

In his "Elogy" on Leibniz, read before the French Academy in 1717, Fontenelle declared in regard to the principle of sufficient reason: "His great principles were that nothing exists or is accomplished without a sufficient reason; that these changes are not made abruptly and by jumps, but by degrees and shadings ["nuances"] as in sequences of numbers or in curves; . . . .[10]

8. *Metaphysics: the Problem of Ultimate Reality.* In the treatment of metaphysics, one of Leibniz's principal points was his doctrine of *monads.* A monad was a metaphysical point, a center of force. According to Erdmann,

The process by which all existence is contained in the single monad Leibnitz has described. . . . It is especially in his correspondence with Arnauld that he tries to explain it. Just as the centre of a circle is the meeting place of all the radii, and therefore con-

[8] Bertrand Russell, *A Critical Exposition of the Philosophy of Leibniz* (London: G. Allen and Unwin, 1937), pp. 169, 170. For Leibniz's statement quoted here by Russell, see Erdmann, 1840, p. 83.

[9] Erdmann, *op. cit.* 1840, p. 748.

[10] "Eloge de Leibnitz par M. de Fontenelle," Bruyset, *op. cit.*, I, 43.

tains all the central angles, so the monad contains all or expresses everything. He puts the matter in the same way against Bayle.

Leibnitz is as emphatic as the atomists in maintaining the indivisibility of his monads; but while the atoms, as being extended, remain divisible at least in thought, the monads, like mathematical points, are actually indivisible, and they are distinguished from the atoms by being not merely modalities, but something real. They are therefore metaphysical points.[11]

According to Carr, Leibniz sought to extend analysis to metaphysics when he declared: "The art of proving metaphysical propositions, ... demands extreme precautions and a greater precision even than those required in mathematics."[12]

According to Couturat:

.... it is necessary to render an exact account of what he wished to say when he considered mathematical and mechanical law as insufficient to explain the universe. Metaphysical principles are neither opposed to nor added to the laws of mathematics; they are superimposed on these laws.[13]

The methods of geometers always appeared to Leibniz to be the ideal and universal method, the best guarantee of logical correctness; and he criticized Descartes and Spinoza not because they did not use it outside of mathematics but because they used it poorly. We know that he flattered himself for succeeding better than they did, thanks to his universal characteristic which should give to all the sciences the rigor and precision of mathematics. That is why he is going to say, "My metaphysics is all mathematical, actually or potentially.[14]

9. *Natural Philosophy: the Problem of the External World.* The importance which Leibniz gave to mathematics in dealing with the problem of the external world was revealed in an "Epistle" to M. Huet, in which he wrote:

Some authors, famous for beautiful discoveries and excellent systems, turned their minds toward the study of nature, hoping that with the aid of mathematics they would succeed in knowing it. These authors were Galileo in Italy; Bacon, Harvey, and Gilbert in England; Descartes and Gassendi in France; .... It is necessary to admit that this kind of study has great attraction and evident utility.[15]

The most important point to be mentioned in considering the influence of mathematics on the natural philosophy of Leibniz is

[11] Erdmann, *op. cit.* 1924, p. 178.

[12] H. W. Carr, *Leibniz* (Boston: Little, Brown, and Company, 1929), p. 30.

[13] Louis Couturat, *La Logique de Leibniz* (Paris: F. Alcan, 1901), p. 238.

[14] *Ibid.*, p. 280.

[15] Bruyset, *op. cit.*, I, 101.

his theory that the changes in nature were infinitesimally small changes. This theory was rather closely related to his concept of the infinitesimal calculus. He wrote in the *Journal des Savans* in 1696:

> I really believe that matter is essentially an aggregate, and as a consequence, that it is composed of real parts. It is by the reason and not only by the senses, that we conclude that it is divided, or better, that it is originally an aggregate. I believe that it is true that matter (and likewise each part of matter) is divided into a larger number of parts than can be imagined. That is what causes me to say that each body, however small it may be, is a world of an infinite number of creatures.[16]

Latta declared that the influence of mathematics on the philosophy of Leibniz appeared chiefly in connection with his law of continuity.[17] Just as the volume of a solid could be found by integration, so could the entire external world be considered as an aggregate of infinitesimals. Latta made the following statement in regard to Leibniz's assertion in the *Monadology* that compounds were analogous ["symbolisent avec"] with simple substances:

> The expression 'symbolize' suggests the 'calculus' idea which is so continually in Leibniz's mind. As numbers are symbols of the things numbered, and we make accurate calculations without referring at every step to the particular things for which our symbols stand, so in general unanalyzed thoughts may be symbols of their simple elements. In the same way compound things are symbols of the simple substances which compose them. What is perceived confusedly in compounds is not a mere illusion but an imperfect representation or symbol of the real characteristics of simple substances. Thus, in this section, Leibniz would say that the spatial or material *plenum* (which is a confused perception of ours) is a symbol of the infinite (or perfectly complete) series of Monads, which has no gaps, since the Monads differ from one another by infinitely small degrees. Similarly, the material action and re-action throughout the universe, such that a change at any one point affects every other, is a symbol of the Pre-established Harmony among the Monads.[18]

Another point which may be mentioned in connection with Leibniz's treatment of natural philosophy was his theory of the conservation of *force* as opposed to Descartes' earlier theory of the conservation of *motion*. Leibniz was thus a step closer to the law of the conservation of energy. His point of view in this regard was related to mathematics. According to Höffding:

[16] Erdmann, *op. cit.* 1840, p. 135.

[17] Robert Latta, *Leibniz; the Monadology* (Oxford: Oxford University Press, 1898), p. 83.

[18] *Ibid.*, p. 251 fn.

In a treatise on Natural Philosophy (*Hypothesis Physica Nova,* 1671) he, like Hobbes and Gassendi, asserts the continuity of motion by means of the idea of tendencies to motion (conatus) throughout the smallest moments and in the smallest parts. This idea is connected with the discovery made in the following year of the significance of infinitely small quantities in mathematics. From this time forward he protests against Descartes' doctrine of the conservation of motion, and begins to assert in its stead the conservation of force.[19]

10. *Practical Philosophy: the Problems of Ethics, Esthetics, and Politics.* Leibniz wrote extensively in ethics and politics, but he wrote very little that could be called esthetics. Mathematics was related to his treatment of ethics, for he thought that men could reason in ethics, like metaphysics, very much as they did in geometry. Characters should be used to fix ethical ideas which would otherwise be too vague and fleeting. In a letter to Montmort, Leibniz compared the points of inflection of a mathematical function to the crises in one's life, writing:

> Just as in a line of geometry there are certain points, distinguishable from each other such as . . . . points of inflection, and as there are lines which are infinitely long; just so it is necessary to conceive in the life of an animal or a person times of extraordinary change which cannot be dealt with by the general rule.[20]

Another point may be mentioned in regard to Leibniz's treatment of ethics. Erhard Weigel, a professor of mathematics at Jena, propounded an ingenious scheme for inculcating morals into children by means of arithmetic (*Arithmetische Beschreibung der Moral Weisheit,* 1674). The scheme won the admiration of Leibniz, who wrote:

> I entirely approve the excellent designs of our Weigel for inculcating useful ideas into children's minds. . . . There is nothing more delightful than the analogies he draws from mathematical things and applies in various ways to moral things; . . . .[21]

When the Germans wanted an excuse to take over Poland in 1669, Leibniz was called upon to write a political treatise showing that the Count Palatine of Neuberg should be King of Poland. Leibniz attempted to show by mathematical demonstration that it was necessary in the interest of Poland to have a German ruler.

[19] Harold Höffding, *A History of Modern Philosophy,* translated by B. E. Meyer (New York: Macmillan and Company, 1900), p. 334.

[20] Bruysett, *op. cit.,* II, 170.

[21] Carr, *op. cit.,* p. 29.

Leibniz wrote the following in regard to his use of mathematics in the political treatise:

I have approved very strongly the thoughts of Mr. Petty on my application of mathematics to matters of economics and politics in a little book printed in the year 1669, on the election of a king of Poland. I made them see that there is a kind of mathematics in the process of reasoning, and that now it is necessary to add, now it is necessary to multiply, in order to get the result. This fact has not been noticed by the logicians. An accomplished theologian who was a Professor of Mathematics, asked me later whether one could write theology using a mathematical method (*Methodo Mathematica*). I replied that one could assuredly do so, and that I have made a beginning on it myself, but that it could not be accomplished without putting philosophy, at least in part, in mathematical order.[22]

11. *Summary and Conclusions.* This paper may be summarized by the following statements:

1. Mathematics was related to all the major problems of philosophy as dealt with by Leibniz.

2. Leibniz sought a universal mathematics, to be expressed in terms of a "universal characteristic," which should enable him to deal with all problems.

3. The "universal characteristic" of Leibniz, though never worked out, was significant in the evolution of symbolic logic.

4. Leibniz's philosophical thought was influenced by his work in the method of the calculus.

[22] *G. W. Leibnitti Opera Omnia, Nunc Primum Collecta* (Geneva: Ludovici Dutens, 1768), VI, 243.

The Teachers' Department

Edited by
WM. L. SCHAAF, JOSEPH SEIDLIN, L. J. ADAMS, C. N. SHUSTER

Elements at Infinity in Projective Geometry

BY N. A. COURT
University of Oklahoma

Historically, projective geometry is an outgrowth and an extension of metrical geometry. It is presented as such in Cremona's *Projective Geometry*, the first textbook on the subject that enjoyed an international circulation. With the work of von Staudt, in the middle of the nineteenth century, there began the attempt to emancipate projective geometry from its dependence upon the metric. This, in turn, led to the use of projective geometry as an example of a mathematical science built up according to the strict precepts of postulational principles. The climax of this tendency, in this country, was reached in the very scholarly two-volume work on the subject by Veblen and Young.

When we offer a group of students a first course in *Projective Geometry*, rather than in the *Foundations of Mathematics*, it is absurd to assume that these students do not know any metrical geometry. On the contrary, it is much more purposeful to build on their past experience, and, what is still more important, to square the new notions introduced with those that the student already possesses. The elements at infinity of projective geometry lead to the first head-on collision between the old and the new. It is incumbent upon the instructor to face the difficulty honestly and to make the adjustment; when left to the student, there is invariably confusion and misundersanding. The nature of the elements at infinity and their relation to metrical geometry must be put into clear relief. What follows is, in substance, the introductory lecture in Projective Geometry, as I have been giving it for the last several years.

Projective Geometry in the plane considers two fundamental

* Paper read before the Mathematical Colloquium, University of Oklahoma, May, 1944.

forms: the range of points, A, B, C, situated on a straight line m, and the pencil of rays a, b, c, ... passing through the same point S. Given the range m, we obtain the pencil S, if we join the points of m to the point S (not on m). Thus through every point of m there will pass a ray of S; conversely, every ray of S passes through a point of m, every ray, that is, but one, namely the ray t passing through S and parallel to m. This is a troublesome exception, worthy of further scrutiny.

The two lines m and t do not have a point in common. Does that mean that they have nothing in common? A line, in addition to the many points that it has, possesses also an additional quality or property which we call "direction." The two lines m and t have this quality in common: they have the same direction. We could therefore make the statement that a ray of the pencil S has either a point or the direction in common with the line m.

The famous postulate of Euclid concerning parallel lines may be stated as follows: Through a given point one and only one line can be drawn having a given direction. Hence the usual statement that "a line is determined by two of its points" may be supplemented to read: "or by one point and the direction of the line." Thus in the determination of a line the direction of the line plays the rôle of a point.

These remarks make it clear that the difficulty we encountered in connection with the range of points and the pencil of lines can readily be removed by identifying "direction" with a point. We can eliminate from our geometrical language the word "direction" and endow the line, in addition to all the "ordinary" points that it has, with a new "extraordinary" point. We will thus be able to make the statement that a line through S meets m in a point. In certain cases we may have to inquire whether the common point is an ordinary or an "extraordinary" point, i.e., whether we are dealing with a case of intersecting lines or of parallel lines. But in general, we will pay no attention to this distinction, not any more than we pay, in algebra, to the question as to whether a is greater than b when we write $a-b$. Our "extraordinary" point is usually called the "point at infinity" of the line. This name is justified on the ground that the point of intersection of a line through S with the line m keeps on receding indefinitely from any fixed point on m (say the foot of the perpendicular from S to m) as the line through S approaches the limiting position of parallelism with the line m. Some authors refer to this point as the "improper" point of the line, while others go to the opposite extreme and call it the "ideal" point.

In space, given a plane μ and a point S, any line v of μ and the point S determines a plane; conversely, every plane passing through S cuts the plane μ along a line, every plane, that is, except one, namely the plane λ through S which is parallel to μ. Here again it is not correct to say that since the planes μ and λ have no line in common, they have nothing in common. The two planes have the same "direction," or let us better say the same "orientation," to avoid overworking the same term and to take advantage of the abundance of words in the English language.

Through a point, one and only one plane can be drawn parallel to a given plane. This proposition may be restated by saying: A plane is determined by a point and the orientation of the plane. On the other hand, a plane is determined by a point and a line. Hence in the determination of a plane, the orientation of the plane plays the same rôle as a line. We can thus eliminate the exception noted, if we drop from our geometric vocabulary the word "orientation" and in its place endow the plane with an "extraordinary" line which we may call the "line at infinity", or the "improper" line, or the "ideal" line of the plane. This convention enables us to say that a plane through S always cuts the plane μ along a straight line. Occasionally we may again have to inquire as to whether this line is a line in the ordinary sense, or the fictitious line, i.e., whether we are considering intersecting planes or parallel planes. But in general we have no concern about this distinction. If it were otherwise, the whole scheme would serve no useful purpose.

To return to plane geometry, the introduction of the point at infinity freed us from a certain embarrassment. But this new point raises troublesome questions of its own. Do the points at infinity form a locus, and if so, what is that locus? The difficulty, however, is more apparent than real. Since every line in the plane has one and only one point at infinity, the locus of these points, if there be such, must be met by every line in the plane in one and only one point; hence that locus can only be a straight line, the "line at infinity" of the plane. This is a very fortunate circumstance, since it happens to agree with the "line at infinity" we attributed to the plane when considering the plane in space. This concordance is further strengthened by the consideration of a line v and a plane u parallel to each other. Through v a plane σ may be drawn parallel to u, and the point of intersection of v and u lies on the line common to σ and u, i.e., the line v meets u on the line at infinity of u.

By analogy with the case of the plane, we may ask, what is the locus in space containing all the lines at infinity of all the planes in

space? The answer is based on the consideration that the locus must be a geometric entity with which every plane in space has a line in common and only one; hence that entity must itself be a plane, the "plane at infinity" of space.

Projective geometry enjoys a considerable advantage from the artifice which identifies the direction of a line with a point and the orientation of a plane with a line. The propositions of projective geometry acquire a simplicity and a generality that they could not otherwise have. Moreover, the elements at infinity give to projective geometry a degree of unification that greatly facilitates the thinking in this domain and offers a suggestive imagery that is very helpful in the acquisition of results. On the other hand, projective geometry stands ready to abandon these fictions whenever that seems desirable, and to express the corresponding propositions in terms of direction of a line and the orientation of a plane, to the great benefit of the science of geometry.

But the suggestive power of words is such that we are tempted to forget the precise and severe limitations under which the elements at infinity have been introduced. We are prone to ascribe to the elements at infinity other properties of points, lines, and planes. To take but one example: one might speculate on the implications of the fact that space is limited by a plane. To declare space limitless and to provide that limitless space with a boundary is sheer contradiction, at least in terms. On top of that, to claim that the statement is justified mathematically is utterly unfair. The plane of projecive geometry has no mystical properties; it is simply a figure of speech, a round-about way of saying that through a given point one and only one plane can be drawn parallel to a given plane. Competent mathematicians do not take the "elements at infinity" of projective geometry for anything more than the convenient fiction that they are, within the limits of applicability of these elements, and do not hesitate to forsake them for something else that may prove to be more convenient under different circumstances. Thus in the theory of inversion, for example, the infinity of the plane is thought of as a point, this concept being very useful in the theory of functions of a complex variable.

A much more serious problem arises when it is attempted to introduce the elements at infinity of projective geometry into the metrical geometry of Euclid. The proximity of the two branches of geometry serves as a powerful temptation. If a line has a point at infinity in projective geometry, why not in Euclidean geometry? An intelligent student of mine in a class in Integral Calculus asked

me recently to prove that two parallel lines meet in a point at infinity. He knew nothing of projective geometry. Indeed, why did not Euclid himself think of the trick? A little reflection will show that the elements at infinity would work havoc with metrical geometry. At every point of a line a perpendicular can be erected and only one. Can a perpendicular be erected at the point at infinity of the line? Such a perpendicular does not exist, or at best is indeterminate, and the proposition considered loses its generality. Two perpendiculars to the same line are parallel. If two parallel lines have a point at infinity in common, this contradicts the fundamental proposition that from a point outside a given line one and only one perpendicular can be drawn to the line. The points at infinity would ruin the entire theory of congruence of triangles. What would be the distance between two points on the line at infinity? If the answer is to be infinity, then every point of the line would be equidistant from all the other points on the line. And so on.

From the point of view of metrical geometry the difference between two lines having a common point and two lines having a common direction is fundamental. Two intersecting lines have no common perpendicular, two parallel lines have an infinite number of them. Propositions valid in one case do not hold in the other. The difference in point of view, the difference in interests of the two geometries, make the device eminently useful in the one and totally unworkable in the other.

Writers on projective geometry insist that the straight line of projective geometry is the Euclidean straight line with an extra point added. The projective plane and projective space are, in turn, the Euclidean plane and Euclidean space enriched, respectively, by an additional line and an additional plane, just as nowadays, let us say, bread is enriched by added vitamins. These statements, despite their widespread acceptance, are nevertheless misleading. The extra point which projective geometry claims to add to the Euclidean line is the way in which propective geometry accounts for that property of the straight line which Euclidean geometry recognizes as the "direction" of the line. The difference between the Euclidean line and the projective line is purely verbal. The geometric content is the same.

Equally illusory is the difference between the Euclidean plane and the projective plane. The line at infinity of the projective plane is the way in which projective geometry incorporates into its plane geometry the parallelism of the Euclidean plane. The difference in verbiage may be striking, but the geometric substance is the same,

and there is no justification for the claim of propective geometry that its plane is "richer" than the Euclidean plane. The same considerations obtain for the plane at infinity. Euclidean space has its parallel planes. It suits the convenience of projective geometry to change the terminology and refer to this parallelism of Euclid by speaking of a plane at infinity; but such a change in nomenclature does not constitute an increase in geometric content.

The claim of having "enriched" Euclidean space has not led projective geometry to make any unjustifiable use of its truly marvelous elements at infinity. It is nevertheless desirable that we dot our i's and know precisely the origin and relation of these elements in the two geometries. It makes for clearer thinking. It may also help to dispel some of the fog of mathematical mysticism.

Post-War Planning for Mathematical Education

On February 25, 1944, the Board of Directors of the National Council of Teachers of Mathematics created the Post-War Policy Commission for the study and improvement of mathematics programs for the nation's schools in the post-war period. The first results of the Commission's thinking are embodied in a brief report which appeared in the May, 1944 issue of the *Mathematics Teacher*, reprints of which may be secured from the office of the *Mathematics Teacher*, 525 West 120 St., New York, 27, N. Y., at 10c a copy. In this report the Commission pointed out that:

(1) the school should insure functional competency in mathematics to all who can possibly achieve it;
(2) we should differentiate on the basis of needs, without stigmatizing any group, and that we should provide new and better courses for that very large part of the schools' population whose mathematical needs are not well met by the traditional sequential courses;
(3) we need a completely new approach to the problem of the so-called slow learner;
(4) the teaching of arithmetic can be and should be improved;
(5) the traditional sequential courses in mathematics should be greatly improved.

The fundamental purposes of the Commission may be briefly summarized as follows: to evaluate current mathematical offerings of the nation's schools from Grade 1 through 14; to utilize the experience derived from the mathematical training programs of the armed forces; to find out to what extent the various phases of these programs have a counterpart in the problems and tasks of civilian life; to focus attention on shortages discovered in the mathematical training of competent young men coming from supposedly good schools; to consider possible remedial measures for improving the low competence in dealing with the fundamentals of arithmetic, as revealed by Army and Navy tests; to explore the possibilities of providing worthwhile and essential mathematics courses for those students who have so often been overlooked or neglected in the preparation of the usual curricula; and to encourage teachers to

become better acquainted with, and to make wider use of multi-sensory aids such as have been developed and extensively used in connection with the mathematical training program of the armed forces.

When originally established, the Post-War Policy Commission consisted of five members; since then the personnel has grown to somewhat more than twice that number. At the present time the Commission is vigorously pursuing its major purposes by means of research, correspondence, personal interviews, and group discussions. The preparation of its second report is progressing rapidly and satisfactorily. It will probably be released in May, 1945, in the *Mathematics Teacher*. This report, somewhat more comprehensive than the first, will also be more concrete and specific in its recommendations. The Commission proposes to set forth certain positive, constructive theses which it is prepared to defend, and which are the result of its sober judgment and careful thinking, based upon the best evidence available. These propositions are not to be regarded in any sense as final or perfect, nor are they offered in a dogmatic spirit. They do, however, reflect the sincere convictions of the group as a whole, and it is earnestly hoped that they will influence the thinking of others for the ultimate good of all concerned.

Specifically, the Second Report of the Post-War Policy Commission will deal with the following problems and issues:

1. Improved Teaching of Arithmetic
2. Functional Competency in Mathematics
3. The Mathematics of the Junior High School
4. The Traditional Sequential Courses in Mathematics
5. Courses Differentiated According to Needs
6. Mathematics in the Small High School
7. Counselling in Connection with Mathematics
8. Multi-sensory Aids in Mathematics
9. Mathematics in the Junior College
10. The Education of Teachers with Respect to Arithmetic and other Branches of Mathematics.

The Commission will at all times be grateful for any suggestions and comments, as well as the reactions of readers to its published reports. Frank criticism as well as helpful ideas will be welcome.

At the present time, the membership of the Commission is comprised of the following: Raleigh Schorling, University of Michigan, (Chairman); William Betz, Specialist in Mathematics, Ro-

chester, N. Y.; William Brownell, Duke University, Durham, N. Carolina; Eugenie C. Hausle, Chairman of the Standing Committee in Mathematics, New York City High Schools; Virgil S. Mallory, State Teachers College, Montclair, N. J.; Mary A. Potter, Supervisor of Mathematics, Racine, Wisc.; William L. Schaaf, Brooklyn College, Brooklyn, N. Y.; Rolland R. Smith, Coordinator of Mathematics, Springfield, Mass.; (Mrs.) Ruth Sumner, Oakland, Calif.; F. Lynwood Wren, George Peabody College for Teachers, Nashville, Tenn.; James H. Zant, Oklahoma A. & M. College, Stillwater, Okla.

—W. L. S.

DELAYED ISSUES UNAVOIDABLE!

A sympathetic colleague recently wrote, "November copy at hand and looks fine. I should think you'd go mad trying to get out a magazine in these days!" Simply and vividly expressed! The quotation and comment are a good introduction to these our **most earnest reminders:**

(1) Will contributors who desire **reprints** of their articles indicate this desire and the number of reprints wanted without **undue waiting?** AFTER the type is thrown in reprints can only be had at extra cost.

(2) At the present time, all the expense of reprints must be on the author. A limited number of copies of the Magazine will be mailed, on request, in lieu of reprints.

(3) All civilian programs, including the publication of scientific journals, are being seriously disrupted by war. Hence our readers are requested, once more, to be patient with situations in which delayed issues of the Magazine are unavoidable.

(4) Kindly help the Magazine and ease the Manager's load by **not** waiting for a statement before remitting renewal check.

S. T. SANDERS.

Brief Notes and Comments

Edited by
MARION E. STARK

11. *Multipliers Instead of Ratios.* The trigonometric functions are usually defined as ratios. From each ratio equation the student is expected to deduce two other relations, thus from

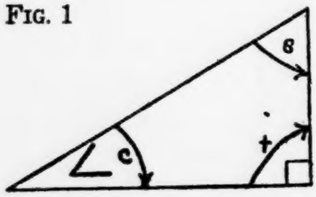
$$\sin A = \frac{a}{c}, \qquad c \sin A = a, \qquad c = a \div \sin A$$

In actual use, these three forms have the relative importance of 1 : 3 : 2. Students trying to get the third in a hurry too often come out with $c = \sin A \div a$. It is much easier and less hazardous to use the middle form as definition and obtain the other two from it. Associated with this proposal is a scheme of marking right triangle diagrams with arrows which make it easy to put down at once the formula wanted instead of a form which is not to be used, but must be transformed first.

This scheme has been used for many years: it works without a hitch for beginners, and almost as well for those who have already been taught the ratio definitions. Only one angle in the right triangle is to be considered. Mark this angle, and draw lines (three of them) to connect pairs of sides. Change these lines to arrows by putting on barbs so that they point

away from the hypothenuse
away from the angle.

FIG. 1



The arrow that points away from *both*, mark "*s*"; the other that points away from the hypothenuse, mark "*c*"; the other that points away from the angle, mark "*t*". Now each arrow represents the number by which you multiply the side from which it springs to get the side to which it points. It follows at once that the side to which it points is divided by the arrow-number to get the side from which it springs. From known to unknown, "multiplication goes with the arrow, division the opposite way." Moreover, a fundamental prop-

erty of the fraction is that the denominator multiplied by the fraction gives the numerator:

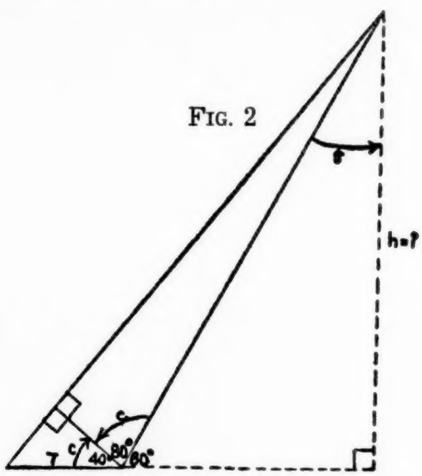
$$b \cdot \frac{a}{b} = a.$$

Hence the arrow-function is read off as the side to which the arrow points over the side from which it springs.

In using this scheme, it is of great importance to consider only *one angle* in each right triangle, and to start always with a *known side*. The scheme becomes exceedingly effective in complicated problems where there is a chain of right triangles. Lines are started from a known side and carried across the several triangles until the unknown is reached. These lines are then barbed and lettered *s, c,* or *t*. This gives in brief form all the necessary formulas for the solution, and one writes them down at once, without further manipulation, in the proper forms for computation.

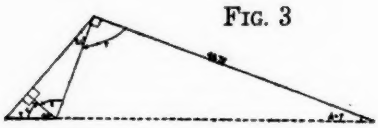
Two examples, of types frequently met, illustrate the complete analysis and formulation of this scheme.

FIG. 2



$$(7 \cdot \cos 40° \div \cos 80°) \sin 60° = h$$

FIG. 3



$$(7 \cdot \cos 40° \div \sin 20°) \div 43.75 = \tan A$$

Tufts College WILLIAM R. RANSOM.

Bibliography and Reviews

Edited by
H. A. SIMMONS and P. K. SMITH

Engineering Problems Illustrating Mathematics. By John W. Cell. McGraw-Hill Co., New York, 1943.

This book is a project of the Mathematics Division of the Society for the Promotion of Engineering Education. It consists of an arrangement of five hundred eleven engineering problems into groups illustrating uses of the subjects of algebra, trigonometry, analytic geometry, differential calculus, and integral calculus, respectively. Answers are supplied. The preface is of more than usual importance in its clear statement of the scope and uses of the book as a teaching aid.

The book is not intended to be a text book. Its proper place is on the teacher's desk if it is to be useful for the purpose stated in the preface of "giving the freshman and sophomore engineering student some understanding of the uses of mathematics in junior and senior engineering courses, and hence of the necessity of a thorough foundation in mathematics."

Most of the problems are designed for special assignment to the better students, though many of them can be solved by the average students. Ingenious statements of the problems furnish the necessary information required to solve problems in specialized applied subjects, so that, while no particular background of training in science is necessary, a student exposed to the material of many of the problems will undoubtedly accumulate an acquaintance with the concepts and notations of engineering subjects which will help him to make more smoothly the change from the study of formal mathematics to the study of engineering subjects. At any rate, every student will profit from the realization that there are other variables beside x and y.

It strikes the reviewer that the material of book will be helpful to many teachers of mathematics in another sense, namely in showing that mathematics can be illustrated in applied form without the teacher trying to carry the burden of teaching the engineering subject to which it is applied.

While perhaps the present-day textbooks in algebra, trigonometry, and analytic geometry are in greater need of the supplementary engineering application than are the textbooks in the calculus, this book will undoubtedly fill an important place in all pre-engineering mathematics courses. The Mathematics Division of the Society for the Promotion of Engineering Education is to be congratulated for the work, and is to be thanked for making it available royalty-free to the teachers who wish to use it.

Louisiana State University. F. A. RICKEY.

A Treatise on the Theory of Bessel Functions. By G. N. Watson. Cambridge, at the University Press; New York, The Macmillan Company. vi+804 p. First Edition, 1922; Second Edition, 1944. Price: $15.00.

One cannot but marvel at the abounding richness and variety of mathematics when he reflects that the contents of this great treatise flow from the functions defined by the linear differential equation:

$$x^2 y'' + xy' + (x^2 - n^2)y = 0.$$

This equation is called the *Bessel equation* and its solutions, *Bessel functions,* after F. W. Bessel (1784-1846), who, although he was not the first writer to use the functions named after him, gave their principal properties and constructed the first tables of $J_0(x)$ and $J_1(x)$ in a lengthy memoir on planetary perturbations published in 1824. As a matter of historical fact, the function of zero order, $J_0(x)$, was used by Daniel Bernoulli in 1732, and the functions of first kind, among functions $J_n(x)$, by L. Euler in 1764 and by J. L. Lagrange in 1769.

The functions thus defined were soon found to have applications in many fields of science. In their varied uses, they rival the circular functions. They are found in physics, astronomy, chemistry, geodesy, geology, and even in a science as remote from their origins as economics. The literature of the subject during the last century has become very voluminous. For example, Watson records in his bibliography 797 books and memoirs published prior to 1922. Some 431 tables of the Bessel functions, and of functions intimately related to them, have been computed, many of them since 1922.

The treatise of Watson has stood for nearly a quarter of a century as the definitive reference work on Bessel functions. The author himself contributed more than a dozen original investigations to the subject. Moreover, from a long series of studies on the general character of asymptotic series, he was peculiarly equipped for an analysis of the intricate problems associated with the asymptotic expanions of the functions and of their zeros. One of the first accounts ever published in English on the powerful *method of steepest descents* (also called the *saddle-point method*) is found in his treatise.

It is unfortunate, however, that this second edition of the classical work on Bessel functions is not a real revision of the first. As the author says: "To incorporate in this work the discoveries of the last twenty years would necessitate the rewriting of at least Chapters XII-XIX; my interest in Bessel functions, however, has waned since 1922, and I am consequently not prepared to undertake such a task to the detriment of my other activities." The revision consists merely of the correction of a few minor errors and the emendation of a few assertions upon which research since 1922 has thrown additional light.

Although one may well regret the fact that the author did not undertake the larger task of revision, it is fortunate that this second edition appears nearly simultaneously with another unusual work on Bessel functions, which to some extent bridges the gap between 1922 and the present time. This is *A Guide to Tables of Bessel Functions* by Harry Bateman and Raymond Claire Archibald, which has just appeared as No. 7 in Vol. I of *Mathematical Tables and other Aids to Computation,* published by the National Research Council. The extensive character of this guide is seen from the fact that, while Watson in Chapter XX devotes 11 pages to the subject of the tabulation of Bessel func-

tions, the report of the Council consists of 104 pages. It lists all tables known to the authors, both published and unpublished, of the Bessel functions, which have been prepared since the first discovery of the functions. It equals in monumental character the work of Watson. Its mass of detailed information gives some indication of the great activity which has centered around the problem of creating adequate tables for the use of the many research workers who employ Bessel functions.

It is needless to say that both the treatise of Watson and the *Guide to Tables* are indispensable works in any mathematical library which attempts to be complete in analysis.

Northwestern University. H. T. DAVIS.

Essentials of Algebra; Complete Second Year Course. By Walter W. Hart. Boston, D. C. Heath and Company, 1943. viii+472 pages. $1.68.

Throughout this review, a number in parentheses after an item indicates approximately its frequency of occurrence; this information may be of interest to some readers. For brevity, the designation, *Complete Course,* is used for the text under review to distinguish it from the author's *Essentials of Algebra, Second Course* of which it is an extension. Only a few (12) of the 339 pages preceding the index of the earlier text are not identical in content with pages of *Complete Course.* Topics taken up in both texts include fundamental operations; factoring; fractions; first and second degree equations and systems; functions, ratio, proportion and variation; square roots and second order radicals; exponents; logarithms; numerical trigonometry; progressions and the binomial theorem; and the graphical solution of cubics. In addition, *Complete Course* has work on radicals; theory of equations; determinants; and very brief sections on the slide rule, mathematical induction, indeterminate forms, algebraic geometry, rates of change, partial fractions, and the geometric addition of complex numbers. The word "Complete" in the title is evidently technical. Some more advanced topics occasionally taken up in courses on college algebra are omitted; among these are inequalities, the general solution of cubics and quartics, and the trigonometric representation of complex numbers. Also, proofs in the chapter on determinants and in that on equations of higher degree are largely omitted. In the opinion of the reviewer, *Complete Course* is not entirely adequate for use in courses in college algebra as they are often taught, but does well what is ordinarily done in the corresponding high school courses.

The arrangement and presentation of material is excellent. The student is guided to a mastery of algebraic techniques and principles by sets (300) of well graded "examples and problems" (6000) for solution, by approximately placed reviews (30) some (12) of which are cumulative or general, and by brief tests (30), about one-third of them diagnostic pre-tests. Student interest is stimulated by numerous solved illustrative examples, carefully constructed graphs (50), general figures (75) and diagrammatic pictures (12), and interesting historical anecdotes (6). For convenient use, tables (4) of second and third powers and roots of integers, logarithms of numbers, and values and logarithms of trigonometric functions, are printed on pages (13) just before the index containing references (500) to pages.

Technical features include unusually careful pagination—starting each new

topic at the top of a left hand page "so that all instruction and initial practice can appear on that page and the one facing it"—confining each set of exercises to one page almost always. New Chapters (18) are sometimes headed by suggestive pictures (7) or faced by framed motivational printed material (4).

The following comments refer specifically to the text. The rule that multiplication precedes division (p. 8) is not agreed upon universally. The deletion of the star indicating that the laws of multiplication (p. 6) are optional material seems apropriate; the reverse indication for binomials used as monomials (p.34) seems questionable. The nature of prime factors (p. 32) is not made entirely clear. The notation used throughout in explaining what is to be done (e.g. p. 65) including: R. P. for remove parentheses, A_{120} for add 120, D_{-7} for divide by -7 etc., is brief and clear. That "only quantities of the same kind can be compared" (p. 97) is controversial. That a point "is the common solution" of two equations (p. 113, 1. 11) is not quite accurate. Pages 154 and 155 should preferably be interchanged. The summary in the classification of numbers (p. 190) is somewhat incomplete. The graphs (pp. 222-225) make the various instances in the solution of quadratic systems crystal clear. The possibility of inconsistent (or dependent) systems (p. 383, ll. 23-26) in which no two equations are inconsistent (or dependent) is neglected.

Misprints which occur are in no case serious; most of them are incidental to revision. Units in *Second Course* are replaced by Chapters in *Complete Course,* but corresponding changes of context have not been made in more than a dozen places, for example, on pages v, 22, 32, 42, 137, 208-9, 233, 235-6 and 279. A statement in the preface, (p. iv, ll. 2-4), indicates that pages 200-201 should be starred. Replace 201 by 200 (p. 211, l. 1); change $\times$ to $+$ (p. 266, l. 3); change 252 to 264 (p. 267, l. 4); change r to r_1 (p. 329, l. 21) and insert f before (x) (p. 329, l. 22); change page 381 to page 380 (p. 393, Rule 2). Answer books of 87 pages are available either bound with the text or separately.

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AN APOLOGY

Difficulties impossible to be overcome account for the shortage of material in this issue. Every effort will be made to make it up in future issues.

—S. T. S.